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Generation of angular momentum traces and Euler polynomials by recurrence relations

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Abstract. Starting from $\mathrm{Tr}(J_M^2)$, we develop traces of $J_L^{2n}J_M^2$, $n \geq 1$, by means of recurrence relations. We also show that the Euler polynomials $E_{2n}(x)$ can be expressed as $F_n(u)$ where $u = x^2 - x$ and obtain $F_n(u)$ by means of recurrence relations.

1. Introduction

Traces of products of angular momentum matrices were first calculated and tabulated by Ambler *et al* (1962a, b) using conventional angular momentum techniques. Rose (1957a, b) who first attracted the attention of theorists to the study of this problem employed recoupling and graphical methods (Rose 1962). Recently there has been considerable interest in these traces (Witschel 1971, 1975, Subramanian and Devanathan 1974, 1980, Pearce 1976, De Meyer and Vanden Berghe 1978a, b, Kaplan and Zia 1979, Rashid 1979, Ullah 1980a, b, c). Witschel used the coupled boson representation introduced by Schwinger (1965) and operator algebra (Witschel 1971) and the comparison method (Witschel 1975) to evaluate such traces. Using the results of Subramanian and Devanathan (1974, hereafter referred to as I), Pearce (1976) obtained interesting counterexamples to pair correlation monotonicity inequalities for the finite spin Heisenberg model, the spin-$\frac{1}{2}$ XY model and the anisotropic planar classical Heisenberg model. Development of traces of $J_-^k J_z^l J_+^k$ by means of recurrence relations (RR) was first achieved by De Meyer and Vanden Berghe (1978b). The present authors (Subramanian and Devanathan 1980, hereafter referred to as II) have generated $\mathrm{Tr}(J_L^{2p})$ from $\mathrm{Tr}(J_L^{2p-2})$ by means of RR starting from $\mathrm{Tr}(J_L^0) = \mathrm{Tr}(I)$. Rashid (1979) obtained a computationally advantageous expression for $\mathrm{Tr}(J_-^k J_z^l J_+^k)$ which exhibits a natural symmetry under the operation $j \to -(j+1)$. Ullah has looked at the trace problem from the point of view of operator identities (Ullah 1980a) and obtained expressions for angular momentum traces in terms of hypergeometric functions (Ullah 1980b, c). It has been shown that the trace of a product of angular momentum matrices is a polynomial in η, the eigenvalue of the $\boldsymbol{J}^2$ operator (I, Kaplan and Zia 1979, Rashid 1979).

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In this paper we show that $\mathrm{Tr}(J_L^{2p-2}J_M^2)$, L, M and N being any permutation of x, y and z (L, M and N are different), can be developed by means of RR starting from $\mathrm{Tr}(J_L^{2p-4}J_M^2)$, $p \geqslant 2$. As a by-product we obtain $\mathrm{Tr}(J_L^{2p})$. In § 2 we obtain the RR between the coefficients of the trace polynomials. Results on the determination of $\mathrm{Tr}(J_L^{2p-2}J_M^2)$ for $p = 6, 7, 8, 9, 10$ and $\mathrm{Tr}(J_L^{20})$ are presented in § 3.

As in the case of Bernoulli polynomials (Miller 1960, Abramowitz and Stegun 1970) which can be generated by means of RR (see II), we show in § 4 that Euler polynomials $E_{2n}(x)$ (Abramowitz and Stegun 1970) can also be developed by means of RR for $n \geqslant 1$, starting from $E_0(x)$.

2. Recurrence relations for trace polynomials

As proved in I, let

$$\Omega^{-1}\,\mathrm{Tr}(J_L^{2p-2}J_M^2) = F_{p-1}(\eta) = \sum_{i=0}^{p-1} f_i\eta^i \tag{1}$$

$$\Omega^{-1}\,\mathrm{Tr}(J_L^{2p-4}J_M^2) = F_{p-2}(\eta) = \sum_{i=0}^{p-2} g_i\eta^i \tag{2}$$

$$\Omega^{-1}\,\mathrm{Tr}(J_L^{2p}) = G_{p-1}(\eta) = \sum_{i=0}^{p-1} a_i\eta^i \tag{3}$$

$$\Omega^{-1}\,\mathrm{Tr}(J_L^{2p-2}) = G_{p-2}(\eta) = \sum_{i=0}^{p-2} b_i\eta^i \tag{4}$$

The quantities η and Ω are defined by

$$\eta = j(j+1) \qquad \Omega = \eta(2j+1) \tag{5}$$

j being the angular momentum quantum number (in units of $\hbar$). In II we have generated a_n from b_i by means of RR. Now consider $\mathrm{Tr}(J_L^{2p-2}\boldsymbol{J}^2)$. Since $\boldsymbol{J}^2 = J_L^2 + J_M^2 + J_N^2 = \eta I$, I being the unit matrix and $\mathrm{Tr}(J_L^{2p-2}J_M^2) = \mathrm{Tr}(J_L^{2p-2}J_N^2)$, we get

$$2F_{p-1}(\eta) + G_{p-1}(\eta) = \eta G_{p-2}(\eta). \tag{6}$$

Equivalently

$$2F_{p-2}(\eta) = \eta G_{p-3}(\eta) - G_{p-2}(\eta). \tag{7}$$

Using equation (2.7) of II, we get

$$\mathscr{D}G_{p-2}(\eta) = 2(p-1)(2p-3)\eta G_{p-3}(\eta) \tag{8}$$

where the operator $\mathscr{D}$ is given by

$$\mathscr{D} = \eta(4\eta+1)\mathrm{D}^2 + 2(7\eta+1)\mathrm{D} + 6 \qquad \mathrm{D} = \mathrm{d}/\mathrm{d}\eta. \tag{9}$$

It follows from equations (7)-(9) that

$$4(p-1)(2p-3)F_{p-2}(\eta) = [\mathscr{D} - 2(p-1)(2p-3)]G_{p-2}(\eta). \tag{10}$$

Equating corresponding coefficients in equations (6) and (10) we have

$$2f_i + a_i = b_{i-1} \qquad 2f_0 + a_0 = 0 \tag{11}$$

$$4(p-1)(2p-3)g_{i-1} = i(i+1)b_i + [2i(2i+1) - 2(p-1)(2p-3)]b_{i-1}. \tag{12}$$

Using equations (3.3)-(3.5) of II to eliminate a_i from relations (11), we get

$$2(i+1)[(i+2)f_{i+1}+2(2i+3)f_i]$$
$$=(i+1)(i+2)b_i+[2(i+1)(2i+3)-2p(2p-1)]b_{i-1}. \tag{13}$$

The relations (12) and (13) are true for all integral values of i provided we assume that $f_k=0$, $k<0$ or $k>p-1$; g_n, $b_n=0$, $n<0$ or $n>p-2$. Thus with $i=p-1$ in equations (12) and (13) we obtain the leading coefficients of $F_{p-1}(\eta)$ and $G_{p-2}(\eta)$:

$$(2p+1)f_{p-1}=b_{p-2}=(2p-3)g_{p-2}. \tag{14}$$

Knowing these leading coefficients, we can generate f_k (and also b_n) from equations (12) and (13).

We would like to stress that the coefficients f_i cannot be generated directly from g_k without calculating the coefficients b_n. Equations (12) and (13) are of the form

$$\alpha g_{i-1}=qb_i+rb_{i-1} \tag{15}$$

$$\beta f_{i+1}+\gamma f_i=sb_i+tb_{i-1} \tag{16}$$

α, β, γ, q, r, s, t being polynomials in i and p. If $s/q=t/r=\delta$, i.e. if $v=qt-rs=0$, then

$$\beta f_{i+1}+\gamma f_i-\alpha\delta g_{i-1}=0 \tag{17}$$

so that f_i can be directly generated from g_k without finding the coefficients b_n. However, from equations (12), (13), (15) and (16) we have

$$v=-4(p-1)(i+1)[2i-(2p-3)] \tag{18}$$

which becomes zero for (a) $p=1$ which is not allowed since $p\geqslant 2$; (b) $i=-1$ which is unacceptable; (c) $i=p-\frac{3}{2}\neq$ integer. Thus relation (17) does not exist in general for all allowed values of p and i. Hence $\mathrm{Tr}(J_L^{2p-2}J_M^2)$ cannot be generated directly from $\mathrm{Tr}(J_L^{2p-4}J_M^2)$. We have to recursively develop the traces via a two-step process. We may call the relations (12) and (13) cascade recurrence relations: $g_k\to b_n\to f_i$.

Technically, knowing the coefficients of $G_{p-2}(\eta)$, one can calculate $F_{p-1}(\eta)$ (see equation (13)) and from $F_{p-1}(\eta)$ ($=F_{(p+1)-2}(\eta)$) one can obtain the coefficients of $G_{p-1}(\eta)$ (see equation (12)) and so on. It is then a matter of taste where to start. In our approach, described in this section, we generate $\mathrm{Tr}(J_L^{2p-2}J_M^2)$ starting from the lowest member of the same family of traces. However, it is pleasant to note that $\mathrm{Tr}(J_L^{2p})$ and $\mathrm{Tr}(J_L^{2p-2}J_M^2)$ can be generated, in principle, from the simplest relation (see II)

$$\mathrm{Tr}(J_L^0)=\mathrm{Tr}(I)=2j+1. \tag{19}$$

3. Results for $\mathrm{Tr}(J_L^{2p-2}J_M^2)$

Starting from (see I)

$$\mathrm{Tr}(J_M^2)=\Omega/3 \tag{20}$$

we can obtain $\mathrm{Tr}(J_L^{2p-2}J_M^2)$, $p\geqslant 2$. We have retrieved our earlier results (table 1 of I) for $p=2, 3, 4, 5$. Further results for $\mathrm{Tr}(J_L^{2p-2}J_M^2)$ for $p=6, 7, 8, 9, 10$ are presented in table 1. As fringe benefits, we retrieve our earlier results (I, II) for $\mathrm{Tr}(J_L^{2p})$,

Table 1. Coefficients N_i of the trace polynomials $\mathrm{Tr}(J_L^{2p-2}J_M^2) = \Omega\,\Sigma_{i=0}^{p-1}\,N_i\eta^i/D$ and the common denominator D. Coefficients f_i of equation (1) are given by $f_i = N_i/D$. As usual $\eta = j(j+1)$, $\Omega = \eta(2j+1)$.

$p \backslash i$	0	1	2	3	4	5	6	7	8	9	D
6	7601	−20 528	19 135	−8050	1225	210					30 030
7	−9555	26 592	−26 450	12 940	−3339	336	42				8190
8	10 851	−30 768	31 755	−16 682	5050	−882	63	6			1530
9	−3728 695	10 705 024	−11 309 656	6188 560	−2019 724	416 416	−52 528	2800	210		67 830
10	69 669 789	−201 771 312	216 573 575	−121 636 354	41 449 275	−9238 614	1382 106	−131 868	5445	330	131 670

$2p = 4, 6, \ldots, 18$. Also we get

$$\begin{aligned}\mathrm{Tr}(J_L^{20}) = (\Omega/3465)[&165\eta^9 - 2475\eta^8 + 22\,770\eta^7\\ &-155\,100\eta^6 + 795\,795\eta^5 - 2981\,895\eta^4\\ &+7704\,835\eta^3 - 12\,541\,460\eta^2\\ &+11\,000\,493\eta - 3666\,831].\end{aligned} \tag{21}$$

We have applied certain checks to our results. The polynomial

$$F_{p-1}(\eta) = \sum_{i=0}^{p-1} f_i\eta^i \qquad p \geq 2 \tag{22}$$

satisfies

$$(a) \qquad f_{p-1} = (4p^2-1)^{-1} \tag{23}$$

$$(b) \qquad f_0 = -B_{2p} \tag{24}$$

$$(c) \qquad 6F_{p-1}(2) = 1 \tag{25}$$

$$(d) \qquad 3(2^{2p-2})F_{p-1}(\tfrac{3}{4}) = 1. \tag{26}$$

Here B_{2p} are the Bernoulli numbers (cf Miller 1960, Abramowitz and Stegun 1970). Equation (23) follows from equations (14), (2) and (20): $(2p+1)(2p-1)f_{p-1} = (2p-1)(2p-3)g_{p-2} = \ldots = (3)(1)g_0 = 3\Omega^{-1}\,\mathrm{Tr}(J_M^2) = 1$. It follows from equation (14) that $b_{p-2} = (2p-1)^{-1}$ and this result is consistent with equation (4.7) of II. Since $2f_0 + a_0 = 0$ (see equation (11)) and $a_0 = 2B_{2p}$ (see equation (4.8) of II), equation (24) follows immediately. To prove equation (25), we note that $\eta = 2$ when $j = 1$. Now by the Cayley-Hamilton theorem $J_L^3 - J_L = 0$ since $\mu^3 - \mu = 0$, $\mu = 1, 0, -1$ being the eigenvalues of J_L. Hence, by induction, $J_L^{2p-2} = J_L^2$, $p \geq 2$. Therefore $6F_{p-1}(2) = 6\Omega^{-1}\,\mathrm{Tr}(J_L^2 J_M^2) = 1$ as $\eta = 2$ (see table 1 of I). When $j = \frac{1}{2}$, $\eta = \frac{3}{4}$ and $J_L = \sigma_L/2$, σ_L being the Pauli spin matrices. Since $\sigma_L^2 = I$, equation (26) follows easily. We have checked that our results satisfy equations (23)-(26).

Using equations (3.3)-(3.5) of II and by means of induction, one can see that the adjacent coefficients of $G_{p-2}(\eta)$, $p \geq 3$, alternate in sign throughout. It then follows from equation (12) and induction that in $F_{p-2}(\eta)$ the leading coefficient and the next coefficient (in decreasing powers of η) are positive and thereafter the adjacent coefficients alternate in sign. Thus g_i and b_i ($i = 0, 1, 2, \ldots, p-3$; $p \geq 3$) have opposite signs. A glance at table 1 of I and the results of II and this section will testify to the correctness of this fact. Items 27-29 of table 1 of I can now be generated by means of RR.

4. Generation of Euler polynomials by recurrence relations

Bernoulli polynomials and Euler polynomials have strikingly similar properties (Abramowitz and Stegun 1970). As in the case of Bernoulli polynomials (Subramanian 1974) one can prove by induction that

$$E_{2n}(x) = S_n(u) \qquad n = 0, 1, 2, \ldots \tag{27}$$

where

$$u = x^2 - x. \tag{28}$$

The proof involves the following results (Abramowitz and Stegun 1970):

$$\mathrm{d}E_n(x)/\mathrm{d}x = nE_{n-1}(x) \qquad n \geqslant 1 \tag{29}$$

$$E_{2n-1}(\tfrac{1}{2}) = E_{2n}(0) = E_{2n}(1) = 0 \qquad n \geqslant 1. \tag{30}$$

Since $\mathrm{d}f(u)/\mathrm{d}x = (2x-1)\,\mathrm{d}f/\mathrm{d}u$, we get from equations (27)-(29)

$$(4u+1)\,\mathrm{d}^2S_n(u)/\mathrm{d}u^2 + 2\,\mathrm{d}S_n(u)/\mathrm{d}u = 2n(2n-1)S_{n-1}(u) \qquad n \geqslant 1. \tag{31}$$

Let

$$E_{2n}(x) = S_n(u) = \sum_{i=0}^{n} C_i u^i \tag{32}$$

$$E_{2n-2}(x) = S_{n-1}(u) = \sum_{i=0}^{n-1} D_i u^i. \tag{33}$$

It follows from equations (31)-(33) that

$$(a) \qquad C_n = D_{n-1} \tag{34}$$

$$(b) \qquad i(i+1)C_{i+1} + 2i(2i-1)C_i = 2n(2n-1)D_{i-1} \tag{35}$$

$1 \leqslant i \leqslant n-1$, $n \geqslant 2$. Equations (34) and (35) are the RR for the Euler polynomials of even order. As equation (31) contains only the derivatives of $S_n(u)$, the constant term C_0 of $S_n(u)$ cannot be obtained from RR. However, it follows from equation (27) and relations (30) that, for $n \geqslant 1$,

$$S_n(u) = E_{2n}(x) = 0 \qquad x = 0, 1 \tag{36}$$

i.e. when $u = 0$. Therefore $S_n(u)$, $n \geqslant 1$, has no constant term. Thus the RR are complete to generate $E_{2n}(x)$ starting from

$$E_0(x) = 1 = u^0. \tag{37}$$

Since (see equation (29))

$$\begin{aligned} E_{2n-1}(x) &= (2n)^{-1}\,\mathrm{d}E_{2n}(x)/\mathrm{d}x = (2n)^{-1}(2x-1)\,\mathrm{d}S_n(u)/\mathrm{d}u \\ &= (2x-1)T_{n-1}(u) \end{aligned} \tag{38}$$

one can also obtain $E_{2n-1}(x)$, $n \geqslant 1$. Alternatively $T_{n-1}(u)$ can also be directly generated by means of RR following a procedure similar to the one given above. Our results for

Table 2. Coefficients C_i of the Euler polynomials $E_{2n}(x) = S_n(u) = \Sigma_{i=0}^{n} C_i u^i$ obtained from $E_0(x) = 1 = u^0$ with $u = x^2 - x$. When $n \geqslant 1$, $C_0 = 0$.

$n \backslash i$	1	2	3	4	5	6	7	8	9
1	1								
2	−1	1							
3	3	−3	1						
4	−17	17	−6	1					
5	155	−155	55	−10	1				
6	−2073	2073	−736	135	−15	1			
7	38 227	−38 227	13 573	−2492	280	−21	1		
8	−929 569	929 569	−330 058	60 605	−6818	518	−28	1	
9	28 820 619	−28 820 619	10 233 219	−1879 038	211 419	−16 086	882	−36	1

$E_{2n}(x)$, $n = 1, 2, \ldots, 9$, are presented in table 2. They are in a more concise form than those given by Abramowitz and Stegun (1970). It is easily seen from table 2 that when $n \geqslant 2$, $C_1 = -C_2$. This result follows from equation (35) and the fact that D_0, the constant term of $S_{n-1}(u)$, $n \geqslant 2$, is zero (see equation (36)). By induction one can show that the adjacent terms of $S_n(u)$, $n \geqslant 2$, alternate in sign throughout. We note that all the coefficients of $S_n(u)$ given in table 2 are integers. We believe that this is true in general, but we have not proved it. The fact that the leading coefficients of $S_q(u)$, $q \geqslant 0$, are all unity follows from equations (34) and (37).

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